MELLIN TRANSFORMS AND ASYMPTOTICS: FINITE DIFFERENCES AND RICE'S INTEGRALS

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March 27, 1994

ABSTRACT. High order differences of simple number sequences may be analysed asymptotically by means of integral representations, residue calculus, and contour integration. This technique, akin to Mellin transform asymptotics, is put in perspective and illustrated by means of several examples related to combinatorics and the analysis of algorithms like digital tries, digital search trees, quadtrees, and distributed leader election.

INTRODUCTION

The problem of estimating *asymptotically* high order differences of some fixed numerical sequence $\{f_k\}$,

$$D_n[f] = \sum_{k=0}^{n} \binom{n}{k} (-1)^k f_k \tag{1}$$

is delicate: the binomial coefficients get close to 2^n while, for many explicitly given sequences, the differences D_n tend to be polynomially bounded in n, and thus exponentially smaller than implied by the trivial bound

$$|D_n[f]| \leq 2^n \max_{0 \leq k \leq n} |f_k|.$$

There is therefore a phenomenon of *exponential cancellation* inherent in most sums of this type which is bound to resist elementary attempts that rely on an asymptotic evaluation of individual terms of the sequence $\{f_n\}$.

The binomial sums of (1) are naturally basic objects of the calculus of finite differences [16, 22, 23]. They acquired interest in the community of researchers working in the average case analysis of algorithms and data structures after De Bruijn, Knuth, and Rice in the mid 1960's showed their central rôle in the analysis of algorithms and data structures that are based on a binary representation of data. The most famous of the first generation examples comprise radix exchange sort, digital "tries" and digital search trees, for which we refer the reader to Knuth's description in [20]: see pages 131–134 (radix exchange sort) and Exercise 5.2.2-54 page 138 (assigned to S. O. Rice), as well as Section 6.3 (tries, Patricia trees, and digital search trees).

This work was partly supported by the ESPRIT Basic Research Action No. 7141 (ALCOM II).

The basic approach to the asymptotic analysis of sums of the form (1) has become known as the technique of "Rice's integrals". The starting point is the integral representation

$$D_n[f] = \frac{(-1)^n}{2i\pi} \int_{\mathcal{C}} \varphi(s) \frac{n!}{s(s-1)\cdots(s-n)}\, ds. \tag{2}$$

There $\varphi(s)$ is an analytic function that "extrapolates" the number sequence $\{f_k\}$ in the sense that $\varphi(k) = f_k$ for all integers k, and $\mathcal{C}$ is a contour that encircles the real interval $[0, n]$. Such integral representations were known much earlier, as attested by the book of Nörlund [23, Chap. 8]; the merit of Rice and his co–workers has been to show how these forms could be used to analyse asymptotically sums like [20, p. 131–138]

$$U_n = \sum_{k=2}^{n} \binom{n}{k} \frac{(-1)^k}{2^{k-1} - 1},$$

which, in addition to exhibiting exponential cancellation, appear to involve subtle periodic fluctuations of a very small amplitude. The sum U_n is directly related to the expected number of bit comparisons necessary to sort a set of n random bit strings by the radix–exchange algorithm [20, Ex. 5.2.2-38].

The present paper is an expanded and updated version of an unpublished memoir of the authors [12] that was written and distributed around 1983. The subject gained renewed interest as similar and often closely related problems surfaced in diverse areas of the analysis of algorithms like: text searching and string matching, communication protocols, variance analysis of digital structures, suffix trees, index trees, multidimensional search and computational geometry, probabilistic algorithms, *etc.*

The technique of Rice's integrals entertains close ties with Mellin transforms. Asymptotic estimates derive from sweeping over poles (for meromorphic $\varphi(s)$) or circling around algebraic–logarithmic singularities, a feature strongly reminiscent of corresponding Mellin asymptotics; singularities of the extrapolation function then contribute asymptotic terms in direct relation to their real part. In fact, the kernel of Rice integrals reduces for large n to a Mellin kernel, a property explored by Szpankowski in [27]. There are other connections (the Poisson-Mellin-Newton cycle) that are hinted at in [9] and briefly reviewed in Section 6.

The authors are extremely grateful to Helmut Prodinger and Wojciech Szpankowski for their invitation to write down this new version of [12] in which we have added some ideas that had remained partly implicit in our earlier manuscript. Rodney Canfield, by asking the question of the behaviour of the sums $D_n[f]$ for $f_k = k^{-\lambda}$, had provided the initial motivation for [12].

Some of the reasons why sums investigated here are of interest in the analysis of digital structures are surveyed in our paper [13] as well as in the combinatorial synthesis [10]. A follow-up to [12] was written by Szpankowski [27], and Prodinger [26] gives an amusing discussion of the method in comparison with standard Mellin transforms techniques.

In what follows, we emphasize general methodology. Bibliographical indications relative to more recent works are given on the occasion of the examples.

1. Differences and generating functions

The differences of a sequence $\{f_n\}$ are classically defined by $\Delta f_n = f_{n+1} - f_n$. The iterated differences are then expressed by alternating binomial sums. In particular, one has

$$\Delta^n f_0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f_k = (-1)^n D_n[f],$$

in the notation of (1). By a slight abuse of language, the $D_n[f]$ are also referred to as "differences".

The transformation of sequences

$$f_n \mapsto g_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k f_k \tag{3}$$

is an involutive transformation, called the *Euler transformation*, that is closely related to the binomial transform and is best viewed as operating on generating functions, either ordinary or exponential.

Ordinary generating functions. Let

$$F(z) = \sum_{n=0}^{\infty} f_n z^n \qquad \text{and} \qquad G(z) = \sum_{n=0}^{\infty} g_n z^n$$

be the *ordinary generating functions* (ogf's) corresponding to $\{f_n\}$ and $\{g_n\}$. Then (3) translates into

$$G(z) = \frac{1}{1-z} F(-\frac{z}{1-z}), \tag{4}$$

which is also known as the Euler transformation of series. It plays a fundamental rôle in the summation of divergent series [14].

Exponential generating functions. Let

$$f(z) = \sum_{n=0}^{\infty} f_n \frac{z^n}{n!} \qquad \text{and} \qquad g(z) = \sum_{n=0}^{\infty} g_n \frac{z^n}{n!}$$

be the *exponential generating functions* (egf's) corresponding to $\{f_n\}$ and $\{g_n\}$. Then, the transformation (3) translates into

$$g(z) = e^z f(-z). \tag{5}$$

In particular, $g(z)$ is a variant of the *Poisson generating function* (Pgf) of the sequence $\{f_n\}$, which is defined by

$$\widehat{f}(z) = \sum_{n=0}^{\infty} f_n e^{-z} \frac{z^n}{n!}.$$

In effect one has $g(z) = \widehat{f}(-z)$.

High order differences thus present themselves whenever the Euler transformation or the Poisson generating function induce simplifications in difference equations, differential equations, or recurrences. In the analysis of algorithms, such is the case for digital tries, digital search trees, and quadtrees, as illustrated by several of the examples below.

2. The integral representation

The analysis of differences starts with a classical integral representation.

Lemma 1. *Let $\varphi(s)$ be analytic in a domain that contains the halfline $[n_0, +\infty[$. Then, the differences of the sequence $\{\varphi(k)\}$ admit the integral representation*

$$\sum_{k=n_0}^{n} \binom{n}{k} (-1)^k \varphi(k) = \frac{(-1)^n}{2i\pi} \int_{\mathcal{C}} \varphi(s) \frac{n!}{s(s-1)\cdots(s-n)}\, ds, \tag{6}$$

where $\mathcal{C}$ is a positively oriented closed curve that lies in the domain of analyticity of $\varphi(s)$, encircles $[n_0, n]$, and does not include any of the integers $0, 1, \ldots, n_0 - 1$.

Proof. This is a direct application of residue calculus, taking into account contributions of the simple poles at the integers $n_0, \ldots, n$. The integral equals the sum of the residues of the integrand,

$$\operatorname*{Res}_{s=k} \varphi(s) \frac{n!}{s(s-1)\cdots(s-n)} = \frac{(-1)^{n-k} n!}{k!(n-k)!} \varphi(k),$$

for $k = n_0, \ldots, n$. $\square$

The kernel in (6) is also expressible in terms of gamma functions,

$$\frac{n!}{s(s-1)\cdots(s-n)} = \frac{\Gamma(n+1)\Gamma(s-n)}{\Gamma(s+1)} = (-1)^{n-1} \frac{\Gamma(n+1)\Gamma(-s)}{\Gamma(n+1-s)},$$

or equivalently in terms of beta functions,

$$\frac{n!}{s(s-1)\cdots(s-n)} = B(n+1, s-n) = (-1)^{n-1} B(n+1, -s),$$

with $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$.

A function $\varphi(s)$ is said to be of *polynomial growth* in an unbounded domain Ω if it is analytic in Ω and satisfies $|\varphi(s)| = \mathcal{O}(|s|^r)$ for some r as $s \to \infty$ in Ω. We also call r the *degree* of $\varphi(s)$. Then, if $\varphi(s)$ is of polynomial growth (is of finite degree) in the half plane $\Re(s) \geq c$ for some $c < n_0$, we have the alternative representation

$$\sum_{k=n_0}^{n} \binom{n}{k} (-1)^k \varphi(k) = -\frac{(-1)^n}{2i\pi} \int_{c-i\infty}^{c+i\infty} \varphi(s) \frac{n!}{s(s-1)\cdots(s-n)}\, ds, \tag{7}$$

valid for n large enough, namely as soon as $n > r + 1$. This alternative form results from taking as integration contour a large segment of the line $\Re(s) = c$ closed to the right by a large semi–circle, see also the proof of case (ii) of Theorem 2. It relies on the fact that, for fixed n, the kernel of Rice's integral is $O(s^{-n-1})$, as $|s| \to \infty$ The sign change is due to orientation.

3. The rational case

Rational functions are *a priori* of polynomial growth in the complement of some compact set. Thus, in the representation (6), one can enlarge the contour of integration, only taking residues into account. This gives rise to an exact representation.

Theorem 1 (Rational functions). *Let $\varphi(s)$ be a rational function analytic on $[n_0, +\infty[$. Then, except for a finite number of values of n, one has*

$$\sum_{k=n_0}^{n} \binom{n}{k} (-1)^k \varphi(k) = -(-1)^n \sum_s \operatorname{Res}\left[\varphi(s) \frac{n!}{s(s-1)\cdots(s-n)}\right], \tag{8}$$

where the sum is extended to all poles s of $\varphi(s)/((s(s-1)\cdots(s-n))$ not on $[n_0, +\infty[$.

Proof. Use the integral representation (6) of Lemma 1, and take as contour of integration a large circle of radius R centered at the origin that avoids the poles. Then let R tend to $+\infty$. The integral converges to 0 provided $n > r + 1$, with r the degree of $\varphi(s)$. By the residue theorem, the integral which is 0 in the limit equals $D_n[\varphi]$ plus the sum of the residues of (8) at the other poles of the integrand. □

Residues correspond to asymptotic terms in the expansion of the differences as made explicit by Proposition 2 below.

A pole of order r at a point s_0 contributes a term of dominant growth

$$n^{s_0} (\log n)^{r-1}.$$

Since $n^{s_0} = \mathcal{O}(n^{\Re(s_0)})$, a collection of residues arranged in decreasing order of real parts form an *asymptotic expansion* of the differences $D_n[\varphi]$.

To make this precise, a few notations are first needed. The *incomplete Hurwitz zeta function* is defined by

$$\zeta_n(r, \beta) = \frac{1}{\beta^r} + \frac{1}{(1+\beta)^r} + \cdots + \frac{1}{(n-1+\beta)^r}.$$

These quantities thus extend the usual harmonic numbers.

The expressions to be derived also involve a variant of the Bell polynomials (see [3] for the standard form). Let $x_1, x_2, \ldots$ be a collection of indeterminates. The *modified Bell polynomials* $\mathbf{L}_m = \mathbf{L}_m(x_1, x_2, \ldots)$ are defined by

$$\exp\left(\sum_{k=1}^{\infty} x_k \frac{t^k}{k}\right) = 1 + \sum_{m=1}^{\infty} \mathbf{L}_m t^m.$$

The expansion above starts as

$$1 + x_1 t + \left(\frac{x_2}{2} + \frac{x_1^2}{2}\right) t^2 + \left(\frac{x_3}{3} + \frac{x_1 x_2}{2} + \frac{x_1^3}{6}\right) t^3 + \left(\frac{x_4}{4} + \frac{x_1 x_3}{3} + \frac{x_2^2}{8} + \frac{x_2 x_1^2}{4} + \frac{x_1^4}{24}\right) t^4 + \cdots,$$

which fixes the first few values, the general formula being

$$\mathbf{L}_m(x_1, x_2, \ldots) = \sum_{1m_1 + 2m_2 + 3m_3 + \cdots = m} \frac{1}{m_1! m_2! m_3! \cdots} \left(\frac{x_1}{1}\right)^{m_1} \left(\frac{x_2}{2}\right)^{m_2} \left(\frac{x_3}{3}\right)^{m_3} \cdots.$$

Proposition 2. *Let α be a complex number not in $\mathbb{N}$. The quantity*

$$I_n(\alpha) = (-1)^n n! \operatorname*{Res}_{s=\alpha} \left(\frac{1}{(s-\alpha)^r} \frac{1}{s(s-1)(s-2)\cdots(s-n)}\right),$$

with r a positive integer, is expressible in terms of harmonic numbers and modified Bell polynomials as

$$\begin{aligned} I_n(\alpha) &= -\frac{\Gamma(n+1)\Gamma(-\alpha)}{\Gamma(n+1-\alpha)}\mathbf{L}_{r-1}(\zeta_{n+1}(1,-\alpha),\zeta_{n+1}(2,-\alpha),\zeta_{n+1}(3,-\alpha),\ldots) \\ &= -\Gamma(-\alpha)n^{\alpha}\mathbf{L}_{r-1}\left(\log n - \frac{\Gamma'(-\alpha)}{\Gamma(-\alpha)},\zeta(2,-\alpha),\zeta(3,-\alpha),\ldots\right)\cdot\left(1+\mathcal{O}(\frac{1}{n})\right) \\ &= -\Gamma(-\alpha)n^{\alpha}\frac{(\log n)^{r-1}}{(r-1)!}\cdot\left(1+\mathcal{O}(\frac{1}{\log n})\right). \end{aligned}$$

Proof. The residue computation reduces to coefficient extraction:

$$\begin{aligned} I_n(\alpha) &= -n!\,[(s-\alpha)^{r-1}]\frac{1}{(-s)(1-s)\cdots(n-s)} \\ &= -n!\,[s^{r-1}]\frac{1}{(-\alpha-s)(1-\alpha-s)\cdots(n-\alpha-s)} \\ &= -n!\,[s^{r-1}]\exp\left(-\sum_{j=0}^{n}\log(j-\alpha-s)\right) \\ &= \frac{-n!}{(-\alpha)(1-\alpha)\cdots(n-\alpha)}[s^{r-1}]\exp(\sum_{m=1}^{\infty}\zeta_{n+1}(m,-\alpha)\frac{s^m}{m}). \end{aligned}$$

The steps consist in shifting the value of s, expanding logarithms, and exchanging summations. The final form results from the definition of modified Bell polynomials. The approximation follows from the estimate

$$\zeta_{n+1}(1,\beta) = \log n - \frac{\Gamma'(\beta)}{\Gamma(\beta)} + \mathcal{O}(\frac{1}{n}),$$

and from standard estimates of the gamma function [32]:

$$\frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} = n^{\alpha}(1+O(\frac{1}{n})).$$

□

EXAMPLE 1. *Differences of inverse powers and harmonic numbers.* Define the sums

$$S_n(m) = \sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^k}{k^m},$$

for m an integer. For negative m, the $S_n(m)$ are eventually null, the nonzero values reducing, up to sign, to Stirling numbers of the second kind [3].

For positive m, the $S_n(m)$ are amenable to Theorem 1 with $n_0 = 1$ and $\varphi(s) = s^{-m}$. There is only a pole of order $m+1$ at $s = 0$, whose residue is to be evaluated. A simple modification of the computation justifying Proposition 2 is needed since $s = 0$. The sum $S_n(m)$ reduces to a coefficient extraction:

$$S_n(m) = -[s^m]\omega_n(s) \qquad \text{where} \qquad \omega_n(s) = \left((1-\frac{s}{1})(1-\frac{s}{2})\cdots(1-\frac{s}{n})\right)^{-1}.$$

This shows that $-S_n(m)$ is the power-sum symmetric function of degree m of the integer inverses $\frac{1}{1}, \frac{1}{2}, \ldots, \frac{1}{n}$. The coefficient may be evaluated by putting $\omega_n(s)$ under exponential form and appealing to the definition of Bell polynomials. We find

$$\omega_n(s) = \exp\left(\sum_{k=1}^{\infty} \zeta_n(k) \frac{s^k}{k}\right),$$

where the $\zeta_n(k)$ are generalized harmonic numbers (equivalently, incomplete zeta functions):

$$\zeta_n(r) \equiv \zeta_n(r, 1) = \frac{1}{1^r} + \frac{1}{2^r} + \cdots + \frac{1}{n^r}.$$

Corollary 3. *The sum $S_n(m)$ is expressible in terms of generalized harmonic numbers as*

$$-S_n(m) = \sum_{m_1+2m_2+\cdots=m} \frac{1}{m_1! m_2! m_3! \cdots} \left(\frac{\zeta_n(1)}{1}\right)^{m_1} \left(\frac{\zeta_n(2)}{2}\right)^{m_2} \left(\frac{\zeta_n(3)}{3}\right)^{m_3} \cdots,$$

and asymptotically

$$-S_n(m) = \sum_{m_1+2m_2+\cdots=m} \frac{1}{m_1! m_2! m_3! \cdots} (\log n + \gamma)^{m_1} \left(\frac{\zeta(2)}{2}\right)^{m_2} \left(\frac{\zeta(3)}{3}\right)^{m_3} \cdots.$$

The first few values of $S_n(m)$ are

$$\begin{aligned}
-S_n(1) &= \zeta_n(1) = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} = \log n + \gamma + \mathcal{O}(\frac{1}{n}) \\
-S_n(2) &= \frac{1}{2}\zeta_n(1)^2 + \frac{1}{2}\zeta_n(2) = \frac{1}{2}(\log n)^2 + \gamma \log n + \frac{\gamma}{2} + \frac{\pi^2}{12} + \mathcal{O}(\frac{\log n}{n}) \\
-S_n(3) &= \frac{1}{6}(\zeta_n(1))^3 + \frac{1}{2}\zeta_n(1)\zeta_n(2) + \frac{1}{3}\zeta_n(3) \\
&= \frac{1}{6}(\log n)^3 + \frac{\gamma}{2}(\log n)^2 + (\frac{\gamma^2}{2} + \frac{\pi^2}{12}) \log n + \frac{\gamma^3}{6} + \frac{\pi^2 \gamma}{12} + \frac{\zeta(3)}{3} + \mathcal{O}(\frac{\log^2 n}{n}).
\end{aligned}$$

Notice also that the polynomial giving the dominant asymptotic form of $S_n(m)$ can be expressed differently. Let $P_m(u)$ be such that

$$-S_n(m) = P_m(\log n) + \mathcal{O}(\frac{\log^m n}{n}),$$

then, from the classical expansion of the Gamma function [32],

$$P_m(u) = [x^m] \exp\left((u+\gamma)x + \zeta(2)\frac{x^2}{2} + \zeta(3)\frac{x^3}{3} + \cdots\right) = [x^m] e^{ux} \Gamma(1-x),$$

so that

$$-S_n(m) = \frac{1}{m!} \sum_{k=0}^{m} \binom{m}{k} (-1)^k \Gamma^{(k)}(1) (\log n)^{m-k} + \mathcal{O}(\frac{\log^m n}{n}). \tag{9}$$

This form generalizes to nonintegral values of m, see Ex. 6.

For $m = 1$, the identity

$$\sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k-1}}{k} = H_n,$$

with $H_n \equiv \zeta_n(1)$ the more familiar notation for the harmonic numbers, is of course extremely well-known and it surfaces in many problems related to random allocations and the theory of records. It

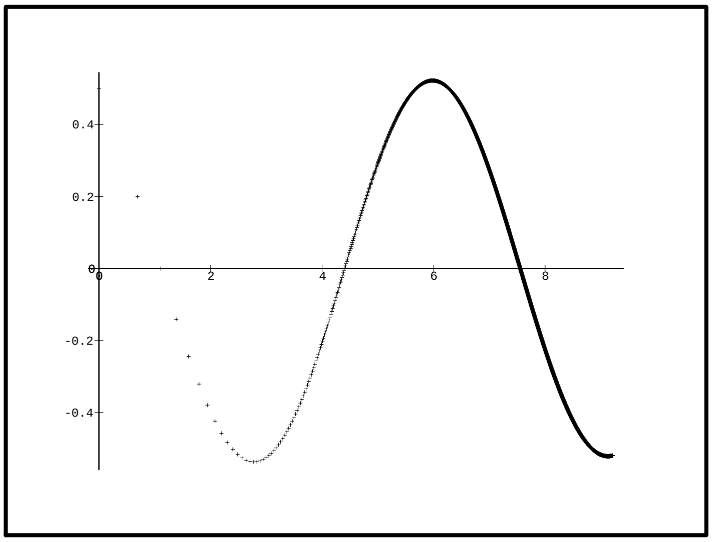

FIGURE 1. A plot of T_n versus $\log n$ for $n \leq 10,000$ displays oscillations that arise from the two complex poles $s = \pm i$ of the function $\varphi(s) = (1+s^2)^{-1}$.

is also related to the exponential integral (by taking exponential generating functions of both sides). More generally, Buchta [2] has shown that $-S_n(m-1)$ equals the expected number of maxima of n vectors in m–dimensional space, a problem of interest in computational geometry, and he has derived second order asymptotics from representations by multiple real integrals of a special type. □

EXAMPLE 2. *A fluctuating function.* The asymptotic analysis of the sequence of numbers

$$T_n = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k^2+1}$$

can be approached from several view points. The sequence is holonomic in the sense of Zeilberger meaning that it satisfies a linear recurrence with polynomial coefficients:

$$T_0 = 1, \qquad T_1 = \frac{1}{2}, \qquad T_n = \frac{n}{n^2+1}((2n-1)T_{n-1} - (n-1)T_{n-2}).$$

This recurrence was communicated to us by Bruno Haible. No elementary asymptotic method seems to be instrumental for estimating directly the rate of growth of this recurrence.

Theorem 1 applies with $\varphi(s) = (1+s^2)^{-1}$, taking the two poles at $s = \pm i$ into account, with residues that are proportional to

$$n^{\pm i} \equiv e^{\pm i \log n}.$$

This provides the asymptotic form

$$T_n = \rho \cos(\log n + \theta_0) + o(1),$$

for two real constants ρ, θ_0 related to $\Gamma(i)$ and $\Gamma(-i)$. In particular, the amplitude ρ is

$$\rho = |\Gamma(i)| = \frac{\pi}{\sqrt{\sinh(\pi)}} \doteq 0.52156.$$

The sequence thus oscillates boundedly, as shown on Figure 1. The sign changes are asymptotically in geometric progression growing roughly in proportion to $N_k = e^{k\pi}$. The T_n thus illustrate some of the peculiarities of the asymptotic analysis of holonomic sequences, of which [33] contains a general discussion. □

This last example demonstrates the fact that *complex poles* of $\varphi(s)$ induce *periodic fluctuations* in $\log n$: if $s_0 = \sigma_0 + i\tau_0$, then

$$n^{s_0} = n^{\sigma_0} \exp(i\tau_0 \log n).$$

4. The meromorphic case

The approach of the last section extends almost *verbatim* to meromorphic functions that are sufficiently well conditioned either on "large contours" or in half–planes.

Theorem 2 (Meromorphic functions). *Let $\varphi(s)$ be a function that is analytic on $[n_0, +\infty[$.*

(i). Assume that $\varphi(s)$ is meromorphic in the whole of $\mathbb{C}$ and analytic on $\Omega = \bigcup_{j=1}^{\infty} \gamma_j$ where the γ_j are positively oriented concentric circles whose radius tends to infinity. Let $\varphi(s)$ be of polynomial growth on Ω. Then, for n large enough,

$$\sum_{k=n_0}^{n} \binom{n}{k} (-1)^k \varphi(k) = -(-1)^n \sum_s \text{Res} \left[\varphi(s) \frac{n!}{s(s-1)\cdots(s-n)} \right],$$

where the sum is extended to all poles s not on $[n_0, +\infty[$.

(ii). Assume that $\varphi(s)$ is meromorphic in the half plane Ω defined by $\Re(s) \geq d$ for some $d < n_0$. Let $\varphi(s)$ be of polynomial growth in the complement in Ω of some compact set. Then, for n large enough,

$$\sum_{k=n_0}^{n} \binom{n}{k} (-1)^k \varphi(k) = -(-1)^n \sum_s \text{Res} \left[\varphi(s) \frac{n!}{s(s-1)\cdots(s-n)} \right] + \mathcal{O}(n^d),$$

where the sum is extended to all poles s in $\Re(s) > d$ and not on $[n_0, +\infty[$.

Proof. In the first case, integrate (6) along the contours γ_j. In the second case, operate with a contour formed of the segment $[d - iR, d + IR]$ closed by the right semi–circle of radius R centered at d. In both cases, take residues into account as in Theorem 1. □

Example 3. *Trie sums.* The prototype of Rice's method that goes back to Rice himself is the treatment of the sum

$$U_n = \sum_{k=2}^{n} \binom{n}{k} \frac{(-1)^k}{2^{k-1} - 1}.$$

Alternating sums like this arise [20, Ex. 5.2.2.36–38] from probabilistic divide-and-conquer recurrences of the form

$$f_n = a_n + 2 \sum_{k=0}^{n} \varpi_{n,k} f_k \qquad \text{where} \qquad \varpi_{n,k} = \frac{1}{2^n} \binom{n}{k}, \tag{10}$$

and they are characteristic of a Bernoulli splitting process. There the a_n normally constitute a simple sequence, and the f_n are to be determined; without loss of generality, we assume that $a_0 = a_1 = 0$. One then introduces the exponential generating functions and the Poisson generating functions: $f(z)$ and $\widehat{f}(z)$ for $\{f_n\}$, $a(z)$ and $\widehat{a}(z)$ for $\{a_n\}$, as defined in Section 1.

The recurrence (10) translates into a functional equation for $f(z)$,

$$f(z) = a(z) + 2e^{z/2} f(\frac{z}{2}),$$

which further simplifies in terms of $\widehat{f}(z)$:

$$\widehat{f}(z) = \widehat{a}(z) + 2\widehat{f}(\frac{z}{2}).$$

This implies for the coefficients, $\widehat{f}_n = n![z^n]\widehat{f}(z)$,

$$\widehat{f}_n = \frac{\widehat{a}_n}{1-2^{1-n}} \qquad \text{so that} \qquad f_n = \sum_{k=2}^{n} \binom{n}{k} \frac{\widehat{a}_k}{1-2^{1-k}},$$

in which the quantities $\widehat{a}_k = k![z^k]\widehat{a}(z)$, are usually simple. The quantity U_n corresponds to the case where $\widehat{a}_k = (-1)^k$ for $k \geq 2$ itself arising from $a_n = (n-1)$ for $n \geq 2$.

The analysis of U_n is a direct application of Theorem 2 when taking as integration contours large circles that go in between the poles of the function $(2^{s-1}-1)^{-1}$. The poles are at

$$\chi_k = 1 + \frac{2ik\pi}{\log 2}.$$

Each of these induces a contribution of the form

$$n^{\chi_k} = ne^{2ik\pi \log_2 n}.$$

The solution appears in Knuth's book [20, Ex. 5.2.2-54], pages 138 and 613–614:

$$\begin{aligned} U_n &= \frac{n}{\log 2}(H_{n-1}-1) - \frac{1}{2}n + 2 + \frac{1}{\log 2} \sum_{k \in \mathbb{Z}\setminus\{0\}} \frac{\Gamma(n+1)\Gamma(-1+\chi_k)}{\Gamma(n+\chi_k)} \\ &= n\log_2 n + \frac{n}{\log 2}(\gamma - 1 - \frac{\log 2}{2}) + \frac{n}{\log 2} \sum_{k \in \mathbb{Z}\setminus\{0\}} \Gamma(-1 - \frac{2ik\pi}{\log 2}) e^{2ik\pi \log_2 n} + \mathcal{O}(\sqrt{n}). \end{aligned}$$

Fluctuations with a similar pattern surface in a great many areas of the analysis of algorithms. Their amplitude is usually $\ll 10^{-5}$ since the gamma function is of fast decrease along the imaginary axis.

From the analysis of U_n and related quantities, one proves that the radix–exchange algorithm sorts n uniformly distributed bit strings in

$$n\log_2 n + nP(\log_2 n) + \mathcal{O}(\sqrt{n})$$

bit comparisons, on average [20]. □

As seen with T_n, poles with nonzero imaginary parts contribute fluctuations that are periodic in $\log n$. In the case of U_n, *regularly spaced poles* (often originating in a periodic meromorphic function) introduce asymptotically a *Fourier series* in $\log n$. As is well known, such behaviours are also captured by Mellin transform techniques whenever the alternating sums can be reorganized as "harmonic sums" [9].

The conditions of Theorem 2 are also susceptible of a large number of variants: rectangular contours may be used instead of circles and contours may be taken that tend to complex infinity in various ways.

EXAMPLE 4. *Probabilistic election of a loser.* This example belongs to the orbit of the so–called "Patricia" variant of digital tries [20, p. 490–504]. Its treatment via Rice's integrals was developed by Prodinger in [25]. The problem involves analysing the sum [25, Theorem 7]:

$$V_n = \sum_{k=1}^{n-1} \binom{n}{k} \frac{B_k}{2^k - 1}.$$

The sum is in fact alternating since the nonzero Bernoulli numbers themselves alternate in sign.

As pointed to us by Peter Grabner (private communication), Prodinger's derivation is correct but the analysis is more subtle than it seems. The corresponding integral representation is

$$V_n = \frac{(-1)^n}{2i\pi} \int_{1/2-i\infty}^{1/2+i\infty} \frac{n!}{(s-1)(s-2)\cdots(s-n)} \frac{\zeta(1-s)}{2^s - 1}\, ds. \tag{11}$$

The main observation is that the Bernoulli numbers are extrapolated by the Riemann zeta function taken at nonnegative integers: $B_k = -k\zeta(1-k)$; the coefficients $(-1)^k$ disappear since the Bernoulli numbers of odd index ≥ 3 are null.

To justify (11), a special argument (provided by P. Grabner), that we now sketch, is needed. The integration contour cannot be extended arbitrarily to the right because of growth properties of $\zeta(s)$ for large negative values of $\Re(s)$: for fixed $\sigma < 0$, one has

$$\zeta(\sigma + it) = \mathcal{O}(t^{1/2-\sigma}) \qquad \text{as } t \to \pm\infty,$$

see [31, p. 95]. One thus integrates along the infinite rectangular contour with vertical sides $\Re(s) = n - 3/4$ and $\Re(s) = 1/2$. The integral along $\Re(s) = n - 3/4$ converges because of growth properties of $\zeta(s)$; it is in fact *identically* zero, a property that generalizes an observation of [6, p. 297] according to which one has

$$0 = \int_{-1/4-i\infty}^{-1/4+i\infty} \zeta(s) \frac{n^s\, ds}{s(s+1)},$$

and is proved by the same means.

Now, the line of integration $\Re(s) = 1/2$ can be moved to the left, with residues being taken into account in the spirit of Theorem 2. The contribution to (11) coming from the residue of the double pole at $s = 0$ is

$$-\frac{1}{\log 2}(H_n - \gamma) + \frac{1}{2} = -\log_2 n + \frac{1}{2} + \mathcal{O}(\frac{1}{n}),$$

and the complex poles at $\chi_k = 2ik\pi/\log 2$ yield a an exact representation in terms of an infinite series, itself asymptotic to a Fourier series:

$$\begin{aligned} V_n &= \frac{H_n - \gamma}{\log 2} - \frac{1}{2} - \frac{1}{\log 2} \sum_{k \in \mathbb{Z}\setminus\{0\}} \frac{\Gamma(n+1)\Gamma(1-\chi_k)}{\Gamma(n+1-\chi_k)} \zeta(1-\chi_k) \\ &= \log_2 n - \frac{1}{2} + Q(\log_2 n) + \mathcal{O}(\sqrt{n}) \text{ where } Q(x) = -\frac{1}{\log 2} \sum_{k \in \mathbb{Z}\setminus\{0\}} \zeta(1-\chi_k)\Gamma(1-\chi_k) e^{2ik\pi x}. \end{aligned}$$

Other examples appear in Prodinger's paper [25] who deduced in this way that a "loser" can be selected probabilistically by means of a tree algorithm on a broadcast network using $\approx \log_2 n$ stages of coin flippings. □

In general, this method may be used to analyse the so–called Patricia tree recurrence, a modified form of the trie recurrence (10):

$$f_n = a_n + \frac{2}{2^n - 2} \sum_{k=1}^{n-1} \binom{n}{k} f_k.$$

Szpankowski *et al.* have extended it to the analysis of tries and Patricia trees under a biased Bernoulli model, see *e.g.*, [18, 28, 29, 30] and references therein. Flajolet and Sedgewick [13], following Knuth, have developed the analysis of digital searching trees in this fashion: difference equations then get replaced by difference–differential equations, with further analyses appearing in [11]. Kirschenhofer, Prodinger, *et al.* in [17, 19] have treated in this way several multidimensional searching problems.

EXAMPLE 5. *Extreme points in quadtrees.* The analysis of the cost of searching points with smallest x–coordinate in a randomly grown quadtree of dimension d calls for estimating the quantity

$$W_n = \sum_{k=2}^{n} \binom{n}{k} (-1)^k [k]\,!,$$

where

$$[k]\,! = \left(1 - \frac{2^{d-1}}{3^d}\right)\left(1 - \frac{2^{d-1}}{4^d}\right) \cdots \left(1 - \frac{2^{d-1}}{n^d}\right), \qquad [2]\,! = 1.$$

The analysis of quadtrees is introduced in Mahmoud [21], and this particular example is borrowed from [7] where cost measures of quadtrees are treated systematically by means of Lindelöf–Mellin integrals and generalized hypergeometric functions. In the case of additive cost measures, the Euler transformation simplifies recurrences by reducing them to first order, as detailed in [7] which demonstrates the relation between W_n and quadtrees.

The problem of analysing W_n then reduces to finding an analytic extrapolation of the sequence $[k]\,!$. By the product formula for the gamma function, one can take

$$\varphi(s) = K \cdot \prod_{\omega^d = 2^{d-1}} \frac{\Gamma(s+1-\omega)}{\Gamma(s+1)} \qquad \text{where} \qquad K = \prod_{\omega^d = 2^{d-1}} \frac{\Gamma(3)}{\Gamma(3-\omega)}.$$

This function is meromorphic in the whole of $\mathbb{C}$ and it remains $\mathcal{O}(1)$ as $\Im(s) \to \pm\infty$ in any right half–plane.

The singularity of largest real part occurs at $s_0 = 2^{(d-1)/d} - 1$. Thus,

$$W_n = K^\star n^{2^{(d-1)/d}-1}\,(1 + o(1)),$$

for some constant $K^\star$ expressible with gamma functions. For instance, when $d = 2, 3, 4$, W_n grows like

$$n^{\sqrt{2}-1} \approx n^{0.41421}, \qquad n^{2^{2/3}-1} \approx n^{0.58740}, \qquad n^{2^{3/4}-1} \approx n^{0.68179}.$$

In [7], other applications are given to path length, search costs, as well as paging constants. □

Apart from quadtrees, the Euler transformation also permits to analyse explicitly generalized digital search trees, as shown in [11]. Thus, Rice's integrals may also be used in this context.

The last two examples also illustrate what is sometimes a nontrivial stage of the method, namely finding a suitable analytic function that extrapolates a given number sequence involving sums or products. The basic principle is as follows. Write $a_k \propto \alpha(s)$ if $\alpha(k) = a_k$ for all integers $k \in \mathbb{N}$, so that $\alpha(s)$ "lifts" the given numerical sequence to the complex. Then, assuming convergence, one has

$$\begin{array}{lll} a_k \propto \alpha(s) & \Longrightarrow & A_n = \prod_{k=1}^{n} a_k \propto A(s) = \prod_{k=1}^{\infty} \frac{\alpha(k)}{\alpha(k+s)} \\ a_k \propto \alpha(s) & \Longrightarrow & A_n = \sum_{k=1}^{n} a_k \propto A(s) = \sum_{k=1}^{\infty} [\alpha(k) - \alpha(k+s)]. \end{array}$$

In general, additional convergence terms must be introduced. See [13] for an application to digital search trees and [17, 18, 19] for applications to analysis of variance and of multidimensional search.

5. The algebraic case

Differences of functions with algebraic or logarithmic singularities are estimated by means of integration contours of Hankel type. The situation is analogous to the asymptotic analysis of coefficients of functions with nonpolar singularities (the method of singularity analysis of [8]), to the application of Mellin–Perron formulæ to Dirichlet series with algebraic or logarithmic singularities [15], or to the analysis of Mellin transforms in the nonpolar case [5].

Rather than stating general conditions that would be rather heavy, we content ourselves here with presenting in some detail the analysis of sums that generalize the $S_n(m)$ when m is no longer an integer. Figure 3 then summarizes the general correspondence between the nature of singularities and the asymptotic form of differences that results.

Example 6. *The differences of $k^{-\lambda}$ for nonintegral λ.* In view of the earlier discussion of the case $\lambda = m$, the resulting sums may be considered as providing harmonic numbers of fractional order. We define:

$$S_n(\lambda) = \sum_{k=1}^{n} \binom{n}{k} (-1)^k k^{-\lambda}.$$

Theorem 3 (An algebraic singularity). *For any non integral λ, the sum $S_n(\lambda)$ has an asymptotic expansion in descending powers of $\log n$ of the form*

$$-S_n(\lambda) \sim (\log n)^{\lambda} \sum_{j=0}^{\infty} (-1)^j \frac{\Gamma^{(j)}(1)}{j!\Gamma(1+\lambda-j)} \frac{1}{(\log n)^j}.$$

This expression therefore generalizes the finite expansion obtained when $\lambda = -m$. We have for instance:

$$\begin{array}{lll} -S_n(-\frac{1}{2}) & = & \frac{1}{\sqrt{\pi \log n}} - \frac{\gamma}{2\sqrt{\pi \log^3 n}} + \mathcal{O}((\log n)^{-5/2}) \\ -S_n(\frac{1}{2}) & = & 2\sqrt{\frac{\log n}{\pi}} + \frac{\gamma}{\sqrt{\pi \log n}} + \mathcal{O}((\log n)^{-3/2}). \end{array}$$

In general the coefficients are expressible rationally in terms of $\gamma, \Gamma(-\lambda)$ and $\zeta(2), \zeta(3), \ldots$. □

Proof. The starting point is again the integral representation

$$(12) \qquad S_n(\lambda) = \frac{1}{2i\pi} \int_{\mathcal{C}} \omega_n(s) \frac{ds}{s^{\lambda+1}} \qquad \text{where} \qquad \omega_n(s) = \left((1-\frac{s}{1})(1-\frac{s}{2})\cdots(1-\frac{s}{n})\right)^{-1}.$$

By Lemma 1 and Equation (7), the integration contour $\mathcal{C}$ may be taken either as the circle of diameter $[\frac{1}{2}, n+\frac{1}{2}]$ or as the vertical line $\Re(s) = \frac{1}{2}$.

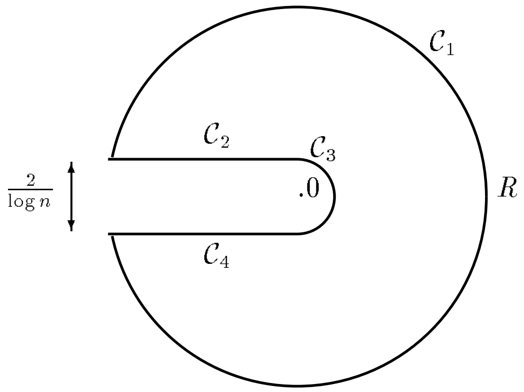

FIGURE 2. The Hankel contour relative to an algebraic or logarithmic singularity (here for $s = 0$).

The basic idea consists in deforming the contour $\mathcal{C}$ so that it extends to the *left* of the singularity at $s = 0$. However, since the singularity is no longer polar, the contour must avoid it. The contour employed resembles the one used in the method of singularity analysis, though the scaling is different: it consists of a loop around the singularity close to it (at a distance of about $1/\log n$) in order to "capture" the main contribution of the algebraic singularity at $s = 0$.

We shall only give the main steps here, since a full proof can be developed along lines quite similar to those of singularity analysis [8]. The estimation relies on the composite contour depicted on Figure 2,

$$\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4,$$

where $R > n$ is a large number and

$$\begin{array}{rcl} \mathcal{C}_1 & = & \{s \;/\; |s| = R,\ |\Im(s)| \geq \frac{1}{\log n} \text{ or } \Re(s) > 0\} \\ \mathcal{C}_2 & = & \{s \;/\; s = (i-t)/\log n,\ t \geq 0,\ |s| \leq R\} \\ \mathcal{C}_3 & = & \{s \;/\; s = e^{i\theta}/\log n,\ \theta \in [-\frac{\pi}{2}, +\frac{\pi}{2}]\} \\ \mathcal{C}_4 & = & \{s \;/\; s = (-i-t)/\log n,\ t \geq 0,\ |s| \leq R\}. \end{array}$$

We decompose the integration path, express in (12) $S_n(\lambda)$ as

$$S_n(\lambda) = J_1 + J_< + J_>,$$

and prove that the main contribution comes from $J_>$ associated to a part of the contour close to $s = 0$.

(i). First, we may let R tend to infinity. The integral J_1 along the outer circle $(\mathcal{C}_1)$ tends to 0 as the integrand there is $\mathcal{O}(R^{-n-\lambda-1})$. In the limit $R = +\infty$, one has

$$J_1 = 0. \tag{13}$$

Thus, it only remains to estimate the contribution due to $\mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4$ with the two branches of $\mathcal{C}_2$ and $\mathcal{C}_4$ now extending all the way towards $-\infty$.

(ii). Let $J_<$ denote the contribution to $S_n(\lambda)$ corresponding to the integral taken along the portion of $\mathcal{C}_2 \cup \mathcal{C}_4$ that is restricted to

$$\Re(s) < -\frac{1}{\sqrt{\log n}}.$$

It turns out that $J_<$ is of a smaller order than any negative power of $\log n$. We prove here the simpler but characteristic estimate:

$$\mu(n) := \int_{t_0}^{+\infty} \frac{dt}{(1+\frac{t}{1})\cdots(1+\frac{t}{n})} = \mathcal{O}(e^{-\frac{1}{2}\sqrt{\log n}}) \qquad \text{where} \qquad t_0 = \frac{1}{\sqrt{\log n}}.$$

The quantity $\mu(n)$ is split into

$$\mu(n) = \mu'(n) + \mu''(n) + \mu'''(n) \qquad \text{where} \qquad \mu'(n) = \int_{t_0}^{1}, \ \mu''(n) = \int_{1}^{n^{1/3}}, \ \mu'''(n) = \int_{n^{1/3}}^{\infty}.$$

First, we have by the unimodality of $\Gamma(s)$ and Stirling's formula

$$\begin{aligned} \mu'(n) &= \int_{t_0}^{1} \frac{n!\Gamma(1+t)}{\Gamma(n+1+t)} dt \\ &= \mathcal{O}(1) \int_{t_0}^{1} n^{-t}\, dt \\ &= \mathcal{O}(e^{-t_0 \log n}) = \mathcal{O}(e^{-1/2\sqrt{\log n}}). \end{aligned}$$

Second, by trivial bounds,

$$\mu''(n) = \int_{1}^{n^{1/3}} \frac{n!}{(1+t)\cdots(n+t)}\, dt = \mathcal{O}(n^{1/3} \times \frac{1}{n}) = \mathcal{O}(n^{-2/3}).$$

Third, by trivial bounds and integration,

$$\mu'''(n) < \int_{n^{1/3}}^{\infty} \frac{dt}{(1+\frac{t}{n})^n} = \mathcal{O}(e^{-\frac{1}{2}n^{1/3}}).$$

Proceeding in the same way with a triple decomposition of $J_<$, one establishes

$$J_< = \mathcal{O}(e^{-\frac{1}{2}\sqrt{\log n}}). \tag{14}$$

(iii). Let $J_>$ denote the contribution given by the integral along the portion $\mathcal{C}^\circ$ of $\mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4$ defined by

$$\Re(s) \geq -\frac{1}{\sqrt{\log n}}.$$

By Stirling's formula the approximation

$$\omega(s) = n^s \Gamma(1-s)(1 + \mathcal{O}(\frac{\log n}{n}))$$

is valid uniformly over $\mathcal{C}^\circ$. Thus,

$$J_> = J^\circ + \mathcal{O}(e^{-\frac{1}{2}\sqrt{\log n}}) \qquad \text{where} \qquad J^\circ = \frac{1}{2i\pi} \int_{\mathcal{C}^\circ} n^s \Gamma(1-s) \frac{ds}{s^{\lambda+1}}. \tag{15}$$

We now perform the change of variable $\zeta = s \log n$, so that the integrand and the contour both get rescaled. This gives

$$J^\circ = (\log n)^\lambda \frac{1}{2i\pi} \int_{\mathcal{D}^\circ} e^\zeta \Gamma(1 - \frac{\zeta}{\log n})\, d\zeta,$$

with $\mathcal{D}^\circ$ the image of $\mathcal{C}^\circ$ by $\zeta = s \log n$. Along $\mathcal{D}^\circ$, one has

$$|\zeta| = \mathcal{O}(\sqrt{\log n}).$$

Thus, one can expand $\Gamma(1-\zeta/\log n)$ into a convergent expansion in powers of $\zeta/\log n$. Interchanging the order of summation and integration, we get

$$J^\circ = (\log n)^{-\lambda} \sum_{m=0}^{\infty} (-1)^m \frac{\Gamma^{(m)}(1)}{m!} \frac{1}{(\log n)^m} \frac{1}{2i\pi} \int_{\mathcal{D}^\circ} e^\zeta \zeta^{m+\lambda-1}\, d\zeta. \tag{16}$$

Let finally $\mathcal{L}$ denote the loop contour obtained by extending the two branches of $\mathcal{D}^\circ$ towards $-\infty$. The completion of $\mathcal{D}^\circ$ into $\mathcal{L}$ in the integral is a classical device of asymptotic analysis (*e.g.*, in Laplace's method); here, it introduces only terms that are smaller than any power of $\log n$. (technically, one must appeal to terminating forms of (16)). On the other hand, the complete integrals along $\mathcal{L}$ have a known expression, by Hankel's representation of the gamma function [32]:

$$\frac{1}{2i\pi} \int_{\mathcal{L}} e^\zeta \zeta^{m+\lambda-1}\, d\zeta = \frac{-1}{\Gamma(1-m-\alpha)}. \tag{17}$$

Thus, by (16), J° admits the asymptotic expansion as stated for $S_n(\lambda)$. By (13), (14), and (15), as well as formulæ (16), (17), the statement follows. □

In summary, we have proved that only a "central" part of the contour matters. This part is small enough, so that the integrand can be locally expanded. Termwise integration after completion of the contour yields the desired expansion.

Example 7. *Another fluctuating sum.* The sum

$$X_n = \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^k}{\sqrt{1+k^2}},$$

is reminiscent of T_n considered above. The extrapolation function now has two branch points at $s = \pm i$ with a local behaviour of the form $(s \pm i)^{-1/2}$, which induces a growth of the form $\sqrt{\log n}$. Thus:

$$X_n = \rho\sqrt{\log n}\cos(\log n + \theta_0) + \mathcal{O}((\log n)^{-1/2}).$$

□

Example 8. *The logarithmic differences and a superexponential jump.* Exactly the same contour integration technique applies to functions with a logarithmic singularity (details omitted). The integral now normalizes to a Hankel integral relative to the derivative of the Gamma function. The result involves iterated logarithms $(\log\log n)$.

Theorem 4 (A logarithmic singularity). *The logarithmic differences*

$$Y_n = \sum_{k=1}^{n} \binom{n}{k} (-1)^k \log n$$

satisfy

$$Y_n = \log\log n + \gamma + \frac{\gamma}{\log n} - \frac{1}{12(\log n)^2}(\pi^2 + 6\gamma^2) + \mathcal{O}(\frac{1}{(\log n)^3}).$$

Singular Part	*Asymptotics*
A singularity of $\varphi(s)$ at $s_0 = \sigma_0 + i\tau_0$	approximately $n^{s_0} = n^{\sigma_0} e^{i\tau_0 \log n}$
Simple pole: $(s - s_0)^{-1}$	$-\Gamma(-s_0)n^{s_0}$
Multiple pole: $(s - s_0)^{-r}$	$-\Gamma(-s_0)n^{s_0}\dfrac{(\log n)^{r-1}}{(r-1)!}$
Algebraic singularity: $(s - s_0)^{\lambda}$	$-\Gamma(-s_0)n^{s_0}\dfrac{(\log n)^{-\lambda-1}}{\Gamma(-\lambda)}$
Logarithmic singularity: $(s - s_0)^{\lambda}(\log(s - s_0))^r$	$-\Gamma(-s_0)n^{s_0}\dfrac{(\log n)^{-\lambda-1}}{\Gamma(-\lambda)}(\log\log n)^r$.

FIGURE 3. A summary of the major correspondences between singular parts of functions at $s_0 \notin \mathbb{N}$ and the asymptotic form of corresponding differences.

A curious form of this result is that the product

$$P_n = \frac{2^{\binom{n}{2}}4^{\binom{n}{4}}6^{\binom{n}{6}}\ldots}{3^{\binom{n}{3}}5^{\binom{n}{5}}7^{\binom{n}{7}}\ldots}.$$

is asymptotic to $e^{\gamma}\log n$. Although the factors grow doubly exponentially, the product only increases logarithmically. □

Entire functions. The situation where $\varphi(s)$ is an entire function can also be treated by the method of Rice integrals in conjunction with the saddle point method. This reflects the corresponding situation for the analysis of inverse Mellin transforms [4]. For instance, the sequence of Kooman and Tijdeman (related to Laguerre polynomials)

$$Z_n = \sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{k!}$$

involves the entire function $\varphi(s) = (\Gamma(s))^{-1}$. This sequence is holonomic as it satisfies the recurrence

$$Z_{n+2} = (2 - \frac{2}{n})Z_{n+1} + (1 - \frac{1}{n})Z_n,$$

and its asymptotic behaviour is of the form

$$Z_n = cn^{-1/4}\sin(2n^{1/2} + \theta) + o(n^{-1/4}),$$

for some constants c, θ. Alternative approaches are discussed in Odlyzko's survey: see [24] and references therein.

6. Mellin transforms and Rice integrals

Rice's integrals entertain close ties with *inverse Mellin transforms* and their *modus operandi* is very similar. We only discuss here the formal ideas underlying the analogy. A variety of sufficient validity conditions are easily supplied.

First, the *the Rice kernel asymptotically reduces to an inverse Mellin kernel.* From the asymptotic form of $\omega_n(s)$, one may expect

$$\frac{1}{2i\pi}\int_{d-i\infty}^{d+i\infty}\varphi(s)\frac{(-1)^n n!}{s(s-1)\cdots(s-n)}\,ds \approx -\frac{1}{2i\pi}\int_{d-i\infty}^{d+i\infty}\varphi(s)\Gamma(-s)n^s\,ds.$$

$$
\begin{tikzcd}
& \{f_n\} \arrow[dl, "\textbf{Poisson GF}"'] & \\
\begin{array}{c}\widehat{f}(t)\\ = \sum f_n e^{-t}\frac{t^n}{n!}\end{array} \arrow[rr, "\textbf{Mellin transform}"'] & & \begin{array}{c}\widehat{f}^{\star}(s)\\ = \int_0^\infty \widehat{f}(t) t^{s-1} dt\end{array} \arrow[ul, "\textbf{Rice's integral}"']
\end{tikzcd}
$$

Figure 4. The Poisson-Mellin-Newton cycle.

This heuristic is discussed by Szpankowski [28] who states some sufficient validity conditions. Note though that its direct application may cause difficulties due to a lack of uniformity of the approximation employed. In this paper, we have preferred to develop a direct approach based on the observation that *residues* of Rice integrals are approximated by *residues* of inverse Mellin integrals at corresponding points.

Another aspect is summarized by the following informal statement:

> **Poisson-Mellin-Newton Cycle.** *The coefficients of a Poisson generating function are expressible as a Rice integral applied to the Mellin transform of the Poisson generating function.*

This observation follows from the computation of the Mellin transform of a Poisson generating function:

$$
\begin{aligned}
\widehat{f}^{\star}(s) &= \int_0^\infty \widehat{f}(t) t^{s-1}\, dt \\
&= \sum_{n=0}^{\infty} \frac{f_n}{n!} \int_0^\infty e^{-t} t^{s+n-1}\, dt \\
&= \Gamma(s)\left[f_0 + f_1 \frac{s}{1!} + f_2 \frac{s(s+1)}{2!} + \cdots\right].
\end{aligned}
$$

The last series is a Newton series whose coefficients are simply recovered by differencing:

$$
f_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \eta(k) \qquad \text{with} \qquad \eta(s) = \frac{\widehat{f}^{\star}(-s)}{\Gamma(-s)}.
$$

The differences are then computable by Rice's integrals. This schema is thus described (at least formally!) by the twin relations

$$
\begin{cases}
f_n = \dfrac{(-1)^n}{2i\pi} \displaystyle\int_{\mathcal{C}} \left(\frac{\widehat{f}^{\star}(s)}{\Gamma(-s)}\right) \frac{n!}{s(s-1)\cdots(s-n)}\, ds \\
\widehat{f}^{\star}(s) = \displaystyle\int_0^\infty \left(e^{-t} \sum_n f_n \frac{t^n}{n!}\right) t^{s-1}\, dt
\end{cases}
$$

This method, nicknamed the "Poisson–Melin–Newton" cycle in [9], is summarized by the diagram of Figure 4.

Tries and digital trees. For instance the treatment of trie sums in Example 3 involves the following Mellin transforms of Poisson generating functions:

$$
\widehat{f}^{\star}(s) = \frac{\widehat{a}^{\star}(s)}{1 - 2^{1+s}}.
$$

Thus, a formal solution to the trie recurrence (10) is

$$f_n = \sum_k \binom{n}{k} \left[\frac{\widehat{a}^\star(-s)}{\Gamma(-s)}\right]_{s=k} \frac{(-1)^k}{1-2^{1-k}}.$$

This "explains" the shape of U_n.

An important class of application is to *digital search trees*, see [13, 20, 21]. There, the basic equations for exponential generating functions are in the form of difference–differential equations

$$\frac{d}{dt} f(t) = a(t) + 2e^{t/2} f(\frac{t}{2}),$$

which simplify under the Poisson generating function:

$$\frac{d}{dt}\widehat{f}(t) + \widehat{f}(t) = \widehat{f}(\frac{t}{2}) + \widehat{a}(t).$$

This last scheme is directly solvable by Mellin transforms,

$$(1-s)\widehat{f}^\star(s-1) + \widehat{f}^\star(s) = 2^s \widehat{f}^\star(s) + \widehat{a}^\star(s),$$

which, being an inhomogeneous difference equation of order 1, admits an explicit solution (as an infinite sum of finite products). We thus have here another general class of problems where the Mellin transform of a Poisson generating function is explicit, so that the Poisson-Mellin-Newton cycle applies. In many ways, this "explains" the success of Rice's method in the analysis of digital trees in [13].

De-Poissonization. The Poisson-Mellin-Newton Cycle is more generally useful for the process called "de-Poissonization" which is involved in recovering coefficients of a generating function from *values* (especially *real* values) of its Poisson generating function whenever enough explicit analytic structure is present. It then permits to justify on such particular examples the Poisson heuristic which reads:

If $\widehat{f}(t)$ is a Poisson generating function

$$\widehat{f}(t) = \sum_{n=0}^{\infty} f_n e^{-t} \frac{t^n}{n!},$$

then, for "smooth" $\{f_n\}$, one has the estimate

$$f_n \sim \widehat{f}(n) \qquad as\ n \to +\infty.$$

The intuition behind this heuristic that is familiar to probabilists is as follows: the Poisson generating function $\widehat{f}(t)$ of a sequence $\{f_n\}$ is a sum of the sequence weighted by the Poisson law of parameter t. For large t, the Poisson law has mean t and standard deviation $\sqrt{t}$. Thus, the weights form predominantly an average of the values of f_n for n near t; if $\{f_n\}$ is known *a priori* to be smooth enough [roughly $f_n \sim f_{n\pm\mathcal{O}(\sqrt{n})}$], then the estimation follows.

The Poisson heuristic thus relates conditionally the value of a function $\widehat{f}(t)$ on the real line to the value of its coefficients. As such, it belongs to the category of Tauberian problems. It was known to Ramanujan who investigated it in some detail [1, p. 57]. For the analysis of digital structures, the process is useful for two related reasons: (i) Poisson generating functions often satisfy functional equations of a simpler form than standard exponential generating functions; (ii) Poisson generating functions normally represent expectations of parameters under a Poisson arrival model, which entails strong probabilistic independence properties.

7. Conclusions.

The method of Rice's integrals is *a priori* useful in applications whenever high order differences occur, a clear source being the Poisson generating function (*cf* the example of trie sums) and the Euler transformation of series (*cf* the example of extreme points in quadtrees). As already mentioned this situation arises frequently in the analysis of digital structures and quadtrees.

The importance of the digital trie model and its variants in the theory of data structures, multidimensional searching, and communication thus justifies in our view to consider the method of Rice's integral that was surveyed here as one of the basic asymptotic techniques of the analysis of algorithms.

Acknowledgements. A question originally posed by Rodney Canfield lead to the first version of this paper. Helmut Prodinger and Wojciech Szpankowski deserve special thanks for inciting us to update [12] and publish it. Bruno Haible kindly provided the interesting example of the fluctuating holonomic sequence. Peter Grabner generously shared his ideas on these questions.

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